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Exercise Set 2

Problems

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3. If two triangles have equal altitudes to one side and the other two sides are correspondingly equal, are they necessarily congruent? Why?

4. Prove that a triangle with two equal medians must be isosceles.

5. Prove that a triangle with two equal altitudes must be isosceles.

6. On the sides OX and OY of $\angle XOY$, take two points A, B and C, D respectively, such that $OA = OC$ and $OB = OD$. Connect AD and BC, which intersect at E. Prove that OE bisects $\angle XOY$.

7. Prove that any side of a triangle is less than half the perimeter of the triangle.

8. Prove that if one side of a triangle equals twice another side, then the length of the shortest side is between $\frac{1}{6}$ and $\frac{1}{4}$ of the perimeter.

9. Prove that the median to any side of a triangle is less than half the sum of the other two sides, but greater than half their difference.

10. Given that in $\triangle ABC$, $AB > AC$, and D is the midpoint of side BC. Prove that $\angle ADB > \angle ADC$ and $\angle BAD < \angle CAD$.

11. Prove that the angle bisector of any angle of a triangle is not greater than the median to the opposite side.

12. Given that in $\triangle ABC$, the angle bisector of $\angle A$ meets side BC at D. Prove that:

(a) $AB > BD$ and $AC > CD$;

(b) If $AB > AC$, then $BD > CD$.

13. Let P be any point inside $\triangle ABC$. Prove that:

(a) $\angle BPC > \angle BAC$;

(b) $\dfrac{1}{2}(BC + CA + AB) < PA + PB + PC < BC + CA + AB$.

14. Let $\triangle ABC$ be isosceles with $AB = AC$. From vertex A, draw a line to a point P on side BC. Through P, draw lines parallel to AC and AB, meeting AB at M and AC at N respectively, forming parallelogram $AMPN$. Prove that all such parallelograms have the same perimeter.

15. Prove that connecting the midpoints of the sides of a quadrilateral in order always forms a parallelogram.

16. Prove that if the two diagonals of a quadrilateral are perpendicular to each other, then the segments connecting the midpoints of opposite sides are equal.

17. Prove that the midpoints of the two legs of a trapezoid and the midpoints of the two diagonals all lie on the same line.

18. Let G be the centroid of $\triangle ABC$. Extend BG and CG to E and F, respectively, such that $GE = 2BG$ and $GF = 2CG$. Prove that E, A, F are collinear.

19. Let O be the center of equilateral triangle $\triangle ABC$. Prove that the perpendicular bisectors of BO and CO trisect side BC.

20. Let I be the incenter of $\triangle ABC$. Through I, draw a line parallel to BC that meets AB, AC at E, F. Prove that $EF = BE + CF$.

21. Let I, I_1, I_2, I_3 be the incenter and the excenters opposite to A, B, C respectively of $\triangle ABC$. Prove that:

(a) $\angle BIC = 90^\circ + \dfrac{1}{2}\angle A$

(b) $\angle BI_1C = 90^\circ - \dfrac{1}{2}\angle A$

(c) $\angle BI_2C = \angle BI_3C = \dfrac{1}{2}\angle A$

22. Let O be the circumcenter of $\triangle ABC$. Prove that $\angle BOC$ equals $2\angle A$ or $360^\circ - 2\angle A$.

23. Let H be the orthocenter of $\triangle ABC$. Prove that $\angle BHC$ equals $180^\circ - \angle A$ or $\angle A$.

24. Prove that if there are four points, and one of them is the orthocenter of the triangle formed by the other three, then each of the other three points has the same property. (Such a set of four points is called an orthocentric system.)