by Shang Qiang · English edition by Jonathan Lam

Exercise Set 2

Problems and Solutions

`Version: a09a68f https://jonathanlamao.com/geometry/`

3. If two triangles have equal altitudes to one side and the other two sides are correspondingly equal, are they necessarily congruent? Why?

Outline

No. Take $\triangle ABC$ with altitude CD to AB, where D lies strictly between A and B and $AD \neq DB$, and let A' be the reflection of A in D, so that A' lies on segment DB.

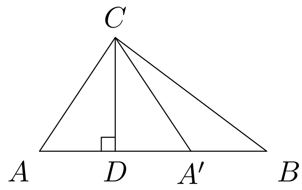

Figure for Problem 3

Compare $\triangle ABC$ with $\triangle A'BC$. Both have the altitude CD from C to the side on line AB. Their other two sides are equal: $CA' = CA$, because CD is the perpendicular bisector of AA', and CB is common. But $A'B = AB - AA' < AB$, so the two triangles are not congruent.

4. Prove that a triangle with two equal medians must be isosceles.

Outline

Let BE and CF be medians of $\triangle ABC$ with $BE = CF$, so E and F are the midpoints of CA and AB. Through F draw the line parallel to BE, meeting line CB at D.

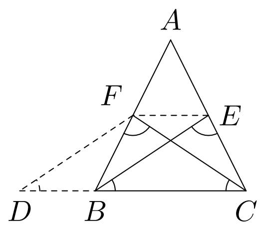

Figure for Problem 4

Since EF is a midline of $\triangle ABC$, $EF \parallel BC$. Thus $EFDB$ has both pairs of opposite sides parallel, so it is a parallelogram, and $FD = BE = CF$. Hence $\triangle FDC$ is isosceles and $\angle FDC = \angle FCD$. Also $\angle FDC = \angle EBC$ (corresponding angles, $FD \parallel EB$), so $\angle EBC = \angle FCB$.

In $\triangle BCE$ and $\triangle CBF$ we now have $BE = CF$, $\angle EBC = \angle FCB$ and the common side BC, so $\triangle BCE \cong \triangle CBF$ (SAS). Therefore $CE = BF$, that is, $\frac{1}{2}CA = \frac{1}{2}AB$, and $AB = AC$.

5. Prove that a triangle with two equal altitudes must be isosceles.

Outline

Let $BE \perp AC$ and $CF \perp AB$ be the two equal altitudes, where E lies on AC and F lies on AB.

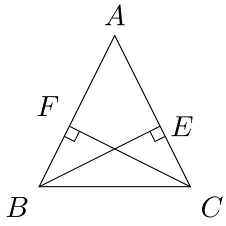

Figure for Problem 5

In right triangles $\triangle BCE$ and $\triangle CBF$:

- $BC = CB$ (common side—the hypotenuse of both right triangles)
- $\angle BEC = \angle CFB = 90^\circ$ (since BE and CF are altitudes)
- $BE = CF$ (given—the altitudes are equal)

By the Hypotenuse-Leg (HL) theorem, $\triangle BCE \cong \triangle CBF$.

Therefore $\angle BCE = \angle CBF$, which means $\angle ACB = \angle ABC$.

Since the base angles are equal, $AB = AC$, and the triangle is isosceles.

6. On the sides OX and OY of $\angle XOY$, take two points A, B and C, D respectively, such that $OA = OC$ and $OB = OD$. Connect AD and BC, which intersect at E. Prove that OE bisects $\angle XOY$.

Outline

Suppose $OA < OB$, so that A lies between O and B and C lies between O and D (the case $OA > OB$ is the same with the roles of the pairs exchanged).

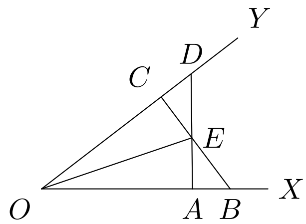

Figure for Problem 6

In $\triangle AOD$ and $\triangle COB$ we have $OA = OC$, $OD = OB$ and the common angle $\angle AOD = \angle COB$, so $\triangle AOD \cong \triangle COB$ (SAS). Hence $\angle OAD = \angle OCB$ and $\angle ODA = \angle OBC$.

Now compare $\triangle EAB$ and $\triangle ECD$. First, $AB = OB - OA = OD - OC = CD$. Next, $\angle EAB = 180^\circ - \angle OAD = 180^\circ - \angle OCB = \angle ECD$. Finally, $\angle EBA = \angle OBC = \angle ODA = \angle EDC$. So $\triangle EAB \cong \triangle ECD$ (ASA), and $EA = EC$.

Finally, $\triangle EOA \cong \triangle EOC$ (SSS: $OA = OC$, $EA = EC$, OE common), so $\angle AOE = \angle COE$, and OE bisects $\angle XOY$.

7. Prove that any side of a triangle is less than half the perimeter of the triangle.

Outline

Let the sides of the triangle be a, b, and c, with perimeter $P = a + b + c$. We prove that $c < \frac{P}{2}$; by symmetry, the same holds for a and b.

By the triangle inequality, $a + b > c$. Adding c to both sides gives $a + b + c > 2c$, that is, $P > 2c$. Dividing by 2, we obtain $c < \frac{P}{2}$.

Therefore any side of a triangle is less than half the perimeter.

8. Prove that if one side of a triangle equals twice another side, then the length of the shortest side is between $\frac{1}{6}$ and $\frac{1}{4}$ of the perimeter.

Outline

As shown in the figure, let $\triangle ABC$ have $AB = 2 \cdot AC$, with $BC = a$ and $AC = b$, so $AB = 2b$. By the triangle inequality, $a = BC > AB - AC = b$, so b is the shortest side.

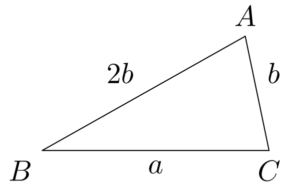

Figure for Problem 8

Let the perimeter of $\triangle ABC$ be l. Then

$$l = AB + BC + CA = 2b + a + b = 3b + a,$$

which gives $a = l - 3b$. By the triangle inequality,

$$AB - AC < BC < AB + AC,$$

that is,

$$b < a < 3b.$$

Substituting $a = l - 3b$ into these bounds,

$$b < l - 3b < 3b.$$

Solving the left inequality $b < l - 3b$ gives $4b < l$, and solving the right inequality $l - 3b < 3b$ gives $l < 6b$. Therefore,

$$\frac{l}{6} < b < \frac{l}{4}.$$

Hence the shortest side b lies strictly between $\dfrac{1}{6}$ and $\dfrac{1}{4}$ of the perimeter.

9. Prove that the median to any side of a triangle is less than half the sum of the other two sides, but greater than half their difference.

Outline

As shown in the figure, let M be the midpoint of BC and N the midpoint of AB. Then MN is a midline of $\triangle ABC$, so $MN \parallel CA$ and $MN = \dfrac{CA}{2}$. Also $AN = \dfrac{AB}{2}$.

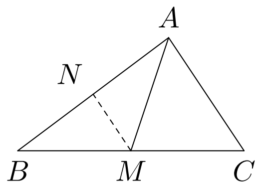

Figure for Problem 9

By the triangle inequality in $\triangle AMN$:

$$AM < AN + MN = \frac{AB}{2} + \frac{CA}{2} = \frac{AB + AC}{2}.$$

Also, $AM > |AN - MN|$, so

$$AM > \left| \frac{AB}{2} - \frac{AC}{2} \right| = \frac{|AB - AC|}{2}.$$

10. Given that in $\triangle ABC$, $AB > AC$, and D is the midpoint of side BC. Prove that $\angle ADB > \angle ADC$ and $\angle BAD < \angle CAD$.

Outline

As shown in the figure, extend AD to E such that $DE = AD$. Since D is the midpoint of BC and $DE = DA$, the diagonals of quadrilateral $ABEC$ bisect each other at D, so $ABEC$ is a parallelogram. Therefore $BE \parallel AC$ and $BE = AC$.

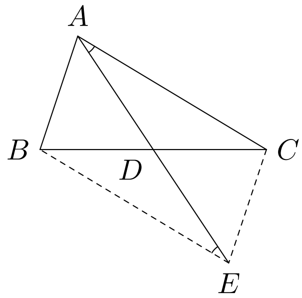

Figure for Problem 10

Second part: Since $AB > AC = BE$, in $\triangle ABE$ the angle opposite AB is larger: $\angle AEB > \angle BAE$. Because $BE \parallel AC$, we have $\angle AEB = \angle CAE = \angle CAD$ (alternate interior angles with transversal AE). Also $\angle BAE = \angle BAD$. Therefore $\angle CAD > \angle BAD$, i.e., $\angle BAD < \angle CAD$.

First part: Since $AB > AC$, the angle opposite AB in $\triangle ABC$ is larger: $\angle C > \angle B$. Adding the inequalities $\angle C > \angle B$ and $\angle CAD > \angle BAD$ gives $\angle C + \angle CAD > \angle B + \angle BAD$. In $\triangle ACD$, $\angle ADB = \angle C + \angle CAD$ (exterior angle), and in $\triangle ABD$, $\angle ADC = \angle B + \angle BAD$ (exterior angle). Therefore $\angle ADB > \angle ADC$.

11. Prove that the angle bisector of any angle of a triangle is not greater than the median to the opposite side.

Outline

(a) When $\triangle ABC$ is isosceles with $AB = AC$, the angle bisector AD and the median AM to side BC coincide (by symmetry), so $AD = AM$.

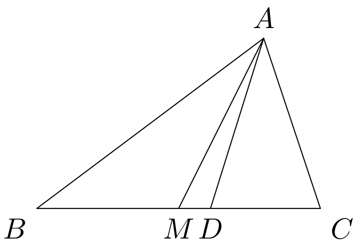

Figure for Problem 11

(b) Suppose $AB > AC$ (the case $AB < AC$ is symmetric). Then $\angle C > \angle B$. By the exterior angle theorem, $\angle ADB = \angle C + \frac{1}{2}\angle A$ and $\angle ADC = \angle B + \frac{1}{2}\angle A$, so $\angle ADB > \angle ADC$. Since these two angles sum to $180°$, $\angle ADB > 90°$.

By Problem 10, $\angle BAM < \angle CAM$, so $\angle BAM < \frac{1}{2}\angle A = \angle BAD$. Hence M lies between B and D. In $\triangle AMD$, the angle $\angle ADM = \angle ADB$ is obtuse, so AM is the longest side, and $AD < AM$.

From (a) and (b), $AD \leq AM$.

12. Given that in $\triangle ABC$, the angle bisector of $\angle A$ meets side BC at D. Prove that:

(a) $AB > BD$ and $AC > CD$;

(b) If $AB > AC$, then $BD > CD$.

Outline

As shown in the figure,

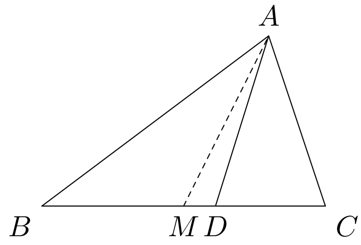

Figure for Problem 12

(a) Since D lies on BC, the angle $\angle ADB$ is an exterior angle of $\triangle ACD$. By the exterior angle theorem, $\angle ADB > \angle DAC$. Since AD bisects $\angle A$, $\angle DAC = \angle DAB$, so $\angle ADB > \angle DAB$. In $\triangle ABD$, the side opposite the larger angle is longer, giving $AB > BD$. By the same reasoning, $\angle ADC > \angle DAB = \angle DAC$, so $AC > CD$.

(b) Draw the median AM to BC. If $AB > AC$, by Problem 11 the angle bisector foot D lies between M and C, so $BD > BM = MC > CD$. (Alternatively, by the angle bisector theorem $BD/CD = AB/AC > 1$.)

13. Let P be any point inside $\triangle ABC$. Prove that:

(a) $\angle BPC > \angle BAC$;

(b) $\frac{1}{2}(BC + CA + AB) < PA + PB + PC < BC + CA + AB$.

Outline

As shown in the figure,

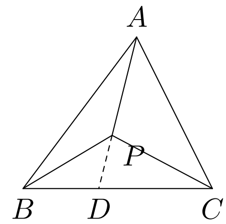

Figure for Problem 13

(a) Extend AP to meet BC at D. In $\triangle ABP$, $\angle BPD$ is an exterior angle, so $\angle BPD > \angle BAP$. In $\triangle APC$, $\angle CPD$ is an exterior angle, so $\angle CPD > \angle CAP$. Adding these two inequalities gives $\angle BPC = \angle BPD + \angle CPD > \angle BAP + \angle CAP = \angle BAC$.

(b) *Lower bound:* By the triangle inequality in $\triangle PAB$, $\triangle PBC$, and $\triangle PCA$:

$$PA + PB > AB, \quad PB + PC > BC, \quad PC + PA > CA.$$

Adding all three gives $2(PA+PB+PC) > AB+BC+CA$, so $PA+PB+PC > \frac{1}{2}(BC+CA+AB)$.

Upper bound: Since P is inside $\triangle ABC$, extend BP to meet AC at a point Q. By the triangle inequality in $\triangle PQC$, $PC < PQ + QC$, so $PB + PC < PB + PQ + QC = BQ + QC$. By the triangle inequality in $\triangle ABQ$, $BQ < BA + AQ$, so $PB + PC < BA + AQ + QC = BA + AC$. Similarly $PA + PC < AB + BC$ and $PA + PB < CA + CB$. Adding:

$$2(PA + PB + PC) < 2(AB + BC + CA),$$

so $PA + PB + PC < AB + BC + CA$.

14. Let $\triangle ABC$ be isosceles with $AB = AC$. From vertex A, draw a line to a point P on side BC. Through P, draw lines parallel to AC and AB, meeting AB at M and AC at N respectively, forming parallelogram $AMPN$. Prove that all such parallelograms have the same perimeter.

Outline

As shown in the figure, since $PM \parallel AC$, in $\triangle BPM$ we have $\angle BPM = \angle BCA$ (corresponding angles). Because $AB = AC$, we have $\angle ABC = \angle BCA$, so $\angle BPM = \angle ABP$. Therefore $\triangle BPM$ is isosceles with $BM = MP$.

Similarly, since $PN \parallel AB$, in $\triangle CPN$ we have $\angle CPN = \angle CBA = \angle ACB$, so $\triangle CPN$ is isosceles with $CN = NP$.

The perimeter of parallelogram $AMPN$ is $AM + MP + PN + NA = AM + BM + CN + NA = AB + AC$, which is constant.

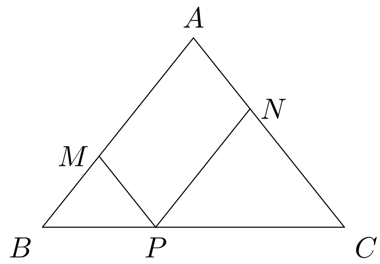

Figure for Problem 14

15. Prove that connecting the midpoints of the sides of a quadrilateral in order always forms a parallelogram.

Outline

As shown in the figure, let E, F, G, H be the midpoints of AB, BC, CD, DA respectively. Draw diagonal AC. In $\triangle ABC$, EF is a midline, so $EF \parallel AC$ and $EF = \dfrac{AC}{2}$. In $\triangle ACD$, HG is a midline, so $HG \parallel AC$ and $HG = \dfrac{AC}{2}$. Therefore $EF \parallel HG$ and $EF = HG$, so $EFGH$ is a parallelogram (one pair of opposite sides is both parallel and equal).

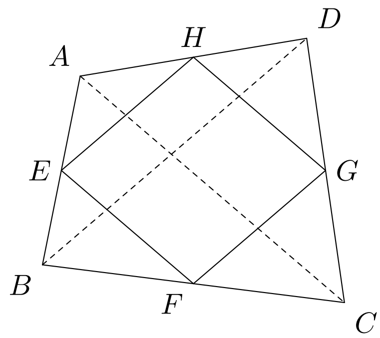

Figure for Problem 15

16. Prove that if the two diagonals of a quadrilateral are perpendicular to each other, then the segments connecting the midpoints of opposite sides are equal.

Outline

As shown in the figure, by Problem 15 the midpoints E, F, G, H of the sides form a parallelogram $EFGH$, with $EF \parallel HG \parallel AC$ and $EH \parallel FG \parallel BD$. Since $AC \perp BD$, we have $EF \perp EH$, so $EFGH$ is a rectangle. The segments connecting midpoints of opposite sides are the diagonals EG and FH of this rectangle, and a rectangle's diagonals are equal, so $EG = FH$.

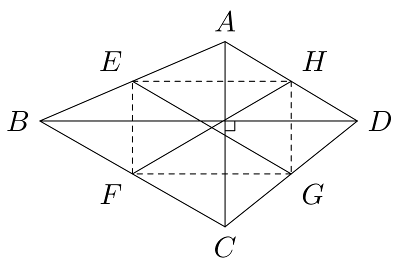

Figure for Problem 16

17. Prove that the midpoints of the two legs of a trapezoid and the midpoints of the two diagonals all lie on the same line.

Outline

As shown in the figure, let $ABCD$ be a trapezoid with $AD \parallel BC$. Let M, N be the midpoints of legs AB, CD, and P, Q the midpoints of diagonals AC, BD.

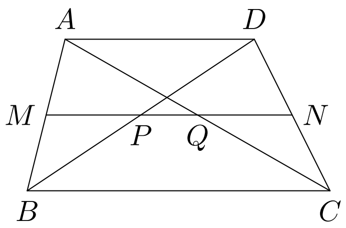

Figure for Problem 17

In $\triangle ABC$: M is the midpoint of AB and P is the midpoint of AC, so MP is a midline with $MP \parallel BC$. In $\triangle ABD$: M is the midpoint of AB and Q is the midpoint of BD, so MQ is a midline with $MQ \parallel AD$. Since $AD \parallel BC$, both MP and MQ are parallel to BC and pass through M, so M, P, Q are collinear.

Similarly, in $\triangle ACD$: $PN \parallel AD$, and in $\triangle BCD$: $QN \parallel BC$. Since $AD \parallel BC$, the points P, Q, N are also collinear on the same line. Therefore M, P, Q, N all lie on one line.

18. Let G be the centroid of $\triangle ABC$. Extend BG and CG to E and F, respectively, such that $GE = 2BG$ and $GF = 2CG$. Prove that E, A, F are collinear.

Outline

As shown in the figure, let M be the midpoint of AC and N the midpoint of AB, so BM and CN are medians meeting at centroid G.

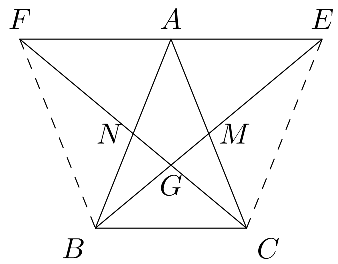

Figure for Problem 18

Since G divides each median in ratio $2:1$ from the vertex, $BG = \frac{2}{3}BM$ and $GM = \frac{1}{3}BM$. Now $GE = 2BG = \frac{4}{3}BM = 4GM$. Therefore $ME = GE - GM = 4GM - GM = 3GM = BM$. Since M is the midpoint of AC and also $BM = ME$ (so M is the midpoint of BE), the diagonals AC and BE of quadrilateral $ABCE$ bisect each other. Hence $ABCE$ is a parallelogram, giving $AE \parallel BC$.

By the same argument applied to median CN: $GF = 2CG = \frac{4}{3}CN = 4GN$, so $NF = GF - GN = 3GN = CN$. Since N is the midpoint of both AB and CF, quadrilateral $ABCF$ is a parallelogram, giving $AF \parallel BC$.

Since $AE \parallel BC$ and $AF \parallel BC$, the points E, A, F are collinear (they lie on the line through A parallel to BC).

19. Let O be the center of equilateral triangle $\triangle ABC$. Prove that the perpendicular bisectors of BO and CO trisect side BC.

Outline

As shown in the figure, let M and N be the midpoints of AB and AC respectively, so BN and CM are medians. In an equilateral triangle, the center O is the centroid, so O lies on median BN with $BO = \frac{2}{3}BN$.

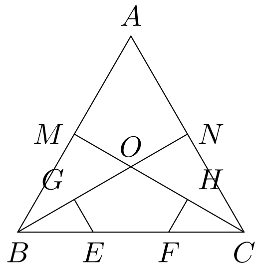

Figure for Problem 19

The midpoint G of segment BO satisfies $BG = \frac{1}{2}BO = \frac{1}{3}BN$. The perpendicular bisector of BO passes through G and is perpendicular to BN. In an equilateral triangle, median BN is also the altitude from B, so $BN \perp AC$. Therefore the perpendicular bisector of BO is parallel to AC.

This perpendicular bisector meets BC at a point E. In $\triangle BNC$, G lies on BN and E on BC with $GE \parallel NC$, so by similar triangles $\frac{BE}{BC} = \frac{BG}{BN} = \frac{1}{3}$, so $BE = \frac{1}{3}BC$. By symmetry, the perpendicular bisector of CO meets BC at a point F with $CF = \frac{1}{3}BC$. Therefore E and F trisect BC.

20. Let I be the incenter of $\triangle ABC$. Through I, draw a line parallel to BC that meets AB, AC at E, F. Prove that $EF = BE + CF$.

Outline

As shown in the figure, since BI bisects $\angle B$ and $EI \parallel BC$, the alternate interior angles give $\angle BIE = \angle IBC = \frac{1}{2}\angle B$. But also $\angle IBE = \frac{1}{2}\angle B$ (since BI bisects $\angle B$). Therefore $\triangle BEI$ is isosceles with $BE = EI$.

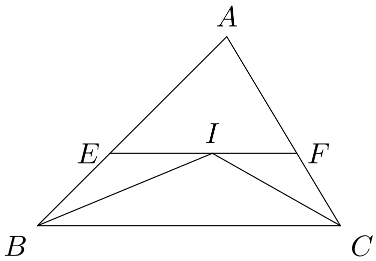

Figure for Problem 20

Similarly, since CI bisects $\angle C$ and $IF \parallel BC$, we get $\angle CIF = \angle ICB = \frac{1}{2}\angle C = \angle ICF$, so $\triangle CIF$ is isosceles with $IF = CF$.

Therefore $EF = EI + IF = BE + CF$.

21. Let I, I_1, I_2, I_3 be the incenter and the excenters opposite to A, B, C respectively of $\triangle ABC$. Prove that:

(a) $\angle BIC = 90^\circ + \dfrac{1}{2}\angle A$

(b) $\angle BI_1C = 90^\circ - \dfrac{1}{2}\angle A$

(c) $\angle BI_2C = \angle BI_3C = \dfrac{1}{2}\angle A$

Outline

As shown in the figure,

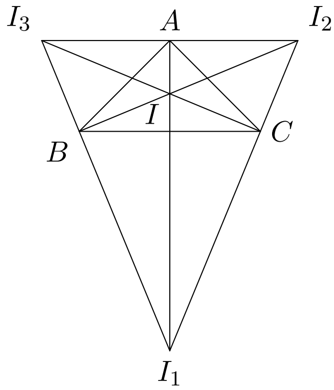

Figure for Problem 21

(a) In $\triangle BIC$, since BI and CI bisect $\angle B$ and $\angle C$:

$$\begin{aligned}\angle BIC &= 180^\circ - \frac{\angle B}{2} - \frac{\angle C}{2} = 180^\circ - \frac{\angle B + \angle C}{2} \\ &= 180^\circ - \frac{180^\circ - \angle A}{2} = 90^\circ + \frac{1}{2}\angle A.\end{aligned}$$

(b) The excenter I_1 opposite A lies at the intersection of the external bisectors of $\angle B$ and $\angle C$. Since BI bisects $\angle B$ internally and BI_1 bisects $\angle B$ externally, $\angle IBI_1 = 90^\circ$. Similarly $\angle ICI_1 = 90^\circ$. Therefore B, I, C, I_1 are concyclic (since $\angle IBI_1 + \angle ICI_1 = 180^\circ$), which gives $\angle BIC + \angle BI_1C = 180^\circ$. Hence

$$\angle BI_1C = 180^\circ - \angle BIC = 180^\circ - 90^\circ - \frac{1}{2}\angle A = 90^\circ - \frac{1}{2}\angle A.$$

(c) The excenter I_2 opposite B lies at the intersection of the internal bisector of $\angle B$ and the external bisectors of $\angle A$ and $\angle C$. So $\angle I_2BC = \frac{1}{2}\angle B$ and the angle $\angle BCI_2 = 90^\circ + \frac{1}{2}\angle C$ (since CI_2 is the external bisector at C). Therefore

$$\angle BI_2C = 180^\circ - \frac{\angle B}{2} - \left(90^\circ + \frac{\angle C}{2}\right) = 90^\circ - \frac{\angle B + \angle C}{2} = \frac{1}{2}\angle A.$$

By symmetry, $\angle BI_3C = \frac{1}{2}\angle A$.

22. Let O be the circumcenter of $\triangle ABC$. Prove that $\angle BOC$ equals $2\angle A$ or $360^\circ - 2\angle A$.

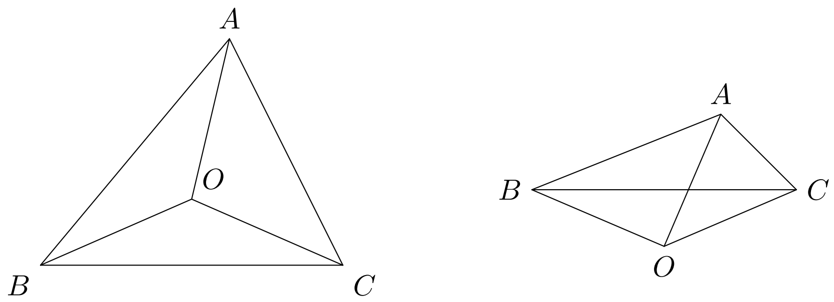

Figure for Problem 22

Outline

(a) When $\angle A$ is acute (left-hand figure), $OA = OB = OC$ gives $\angle OBA = \angle OAB$ and $\angle OCA = \angle OAC$, so $\angle OBA + \angle OCA = \angle A$ and

$$\begin{aligned}\angle BOC &= 180^\circ - (\angle OBC + \angle OCB)\\ &= 180^\circ - [180^\circ - (\angle OBA + \angle OCA + \angle A)]\\ &= 2\angle A\end{aligned}$$

(b) When $\angle A$ is a right angle, BC is a diameter and O is its midpoint, so

$$\angle BOC = 180^\circ = 2\angle A$$

(c) When $\angle A$ is obtuse (right-hand figure), O and A lie on opposite sides of BC and ray OA lies inside $\angle BOC$, so $\angle BOC = \angle AOB + \angle AOC$. Using $\angle OBA = \angle OAB$, $\angle OCA = \angle OAC$ and $\angle OAB + \angle OAC = \angle A$,

$$\begin{aligned}\angle BOC &= [180^\circ - (\angle OBA + \angle OAB)] + [180^\circ - (\angle OCA + \angle OAC)]\\ &= 360^\circ - 2\angle A\end{aligned}$$

23. Let H be the orthocenter of $\triangle ABC$. Prove that $\angle BHC$ equals $180^\circ - \angle A$ or $\angle A$.

Outline

(a) If $\angle A = 90^\circ$, then $H = A$, so $\angle BHC = 90^\circ$, which equals both $\angle A$ and $180^\circ - \angle A$.

(b) If $\angle B = 90^\circ$ or $\angle C = 90^\circ$, then H coincides with B or C, and $\angle BHC$ is not defined.

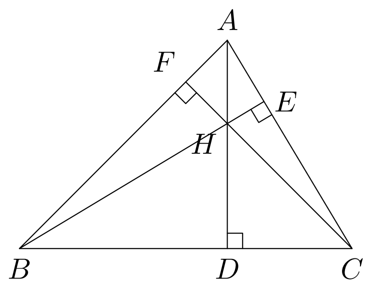

Figure for Problem 23

Otherwise, let E and F be the feet of the altitudes from B and C, so H is the intersection of lines BE and CF, and $\angle HEA = \angle HFA = 90^\circ$ (see the figure).

(c) *The triangle is acute, or $\angle A$ is obtuse.* In quadrilateral $AEHF$ the angles at E and F are right angles, so $\angle EHF = 180^\circ - \angle EAF$. If the triangle is acute, H is inside it, $\angle EAF = \angle A$, and $\angle BHC = \angle EHF$ (vertical angles). If $\angle A$ is obtuse, E and F lie on the extensions of CA and BA beyond A, so $\angle EAF = \angle A$ (vertical angles), and E, F lie on segments HB, HC, so $\angle BHC = \angle EHF$. In both cases $\angle BHC = 180^\circ - \angle A$.

(d) *$\angle B$ is obtuse* (the case of obtuse $\angle C$ is symmetric). Now F lies on the extension of AB beyond B, and H lies on ray CF beyond F and on ray EB beyond B. In right triangle AFC,

$\angle ACF = 90° - \angle A$. In right triangle HEC, $\angle EHC = 90° - \angle ECH = 90° - \angle ACF = \angle A$. Since B lies on segment HE, $\angle BHC = \angle EHC = \angle A$.

24. Prove that if there are four points, and one of them is the orthocenter of the triangle formed by the other three, then each of the other three points has the same property. (Such a set of four points is called an orthocentric system.)

Outline

As in the figure for Problem 23, let H be the orthocenter of $\triangle ABC$. Then $AH \perp BC$, $BH \perp AC$, and $CH \perp AB$.

To show A is the orthocenter of $\triangle HBC$: the altitude from A to BC is $AH \perp BC$ (already true); the altitude from B to HC requires $BA \perp HC$, which holds since $CH \perp AB$; and the altitude from C to HB requires $CA \perp HB$, which holds since $BH \perp AC$. So A is the orthocenter of $\triangle HBC$. By the same reasoning, B is the orthocenter of $\triangle HAC$ and C is the orthocenter of $\triangle HAB$.